Smarandache's Cevian Triangle Theorem in The Einstein Relativistic Velocity Model of Hyperbolic Geometry

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Abstract

In this note, we present a proof of Smarandache's cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry.

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1. Introduction

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis for geometry. It is also known as a type of non-Euclidean geometry, being in many respects similar to Euclidean geometry. Hyperbolic geometry includes such concepts as: distance, angle and both of them have many theorems in common. There are known many main models for hyperbolic geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. The hyperbolic geometry is a non-Euclidian geometry. Here, in this study, we present a proof of Smarandache's cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry. Smarandache's cevian triangle theorem states that if $A_1B_1C_1$ is the cevian triangle of point P with respect to the triangle ABC, then $\frac{PA}{PA_1} \cdot \frac{PB}{PB_1} \cdot \frac{PC}{PC_1} = \frac{AB \cdot BC \cdot CA}{A_1B \cdot B_1C \cdot C_1A}$ [1].

Let D denote the complex unit disc in complex z - plane, i.e.

$$D=\{z\in\mathbb{C}:|z|<1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0}z} = e^{i\theta} (z_0 \oplus z),$$

which induces the Möbius addition \oplus in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0}z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0 . Let $Aut(D, \oplus)$ be the automorphism group

of the grupoid (D, \oplus) . If we define

$$gyr: D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyrocommutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyrovector space (G, \oplus, \otimes) is a gyrocommutative gyrogroup (G, \oplus) that obeys the following axioms:

- (1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.
- (2) G admits a scalar multiplication, \otimes , possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:
 - (G1) $1 \otimes \mathbf{a} = \mathbf{a}$
 - $(G2) (r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$
 - $(G3) (r_1r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$
 - $(G4) \frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$
 - (G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$
 - (G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$
- (3) Real vector space structure ($||G||, \oplus, \otimes$) for the set ||G|| of onedimensional "vectors"

$$||G|| = \{ \pm ||\mathbf{a}|| : \mathbf{a} \in G \} \subset \mathbb{R}$$

with vector addition \oplus and scalar multiplication \otimes , such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

- $(G7) \|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$
- $(G8) \|\mathbf{a} \oplus \mathbf{b}\| \le \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Theorem 1 (The Hyperbolic Theorem of Ceva in Einstein Gyrovector Space) Let $\mathbf{a}_1, \mathbf{a}_2$, and \mathbf{a}_3 be three non-gyrocollinear points in an Einstein gyrovector space (V_s, \oplus, \otimes) . Furthermore, let \mathbf{a}_{123} be a point in their gyroplane, which is off the gyrolines $\mathbf{a}_1\mathbf{a}_2, \mathbf{a}_2\mathbf{a}_3$, and $\mathbf{a}_3\mathbf{a}_1$. If $\mathbf{a}_1\mathbf{a}_{123}$ meets $\mathbf{a}_2\mathbf{a}_3$ at \mathbf{a}_{23} , etc., then

$$\begin{split} \frac{\gamma_{\ominus \mathbf{a}_1 \oplus \mathbf{a}_{12}} \parallel \ominus \mathbf{a}_1 \oplus \mathbf{a}_{12} \parallel}{\gamma_{\ominus \mathbf{a}_2 \oplus \mathbf{a}_{12}} \parallel \ominus \mathbf{a}_2 \oplus \mathbf{a}_{23}} \parallel \ominus \mathbf{a}_2 \oplus \mathbf{a}_{23} \parallel}{\gamma_{\ominus \mathbf{a}_3 \oplus \mathbf{a}_{13}} \parallel \ominus \mathbf{a}_3 \oplus \mathbf{a}_{13} \parallel} = 1, \\ (here \ \gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\parallel \mathbf{v} \parallel^2}{s^2}}} \ is \ the \ gamma \ factor). \end{split}$$
 (see [2, pp 461])

Theorem 2 (The Hyperbolic Theorem of Menelaus in Einstein Gyrovector Space) Let \mathbf{a}_1 , \mathbf{a}_2 , and \mathbf{a}_3 be three non-gyrocollinear points in an Einstein gyrovector space (V_s, \oplus, \otimes) . If a gyroline meets the sides of gyrotriangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ at points \mathbf{a}_{12} , \mathbf{a}_{13} , \mathbf{a}_{23} , then

$$\frac{\gamma_{\ominus \mathbf{a}_1 \oplus \mathbf{a}_{12}} \| \ominus \mathbf{a}_1 \oplus \mathbf{a}_{12} \|}{\gamma_{\ominus \mathbf{a}_2 \oplus \mathbf{a}_{12}} \| \ominus \mathbf{a}_2 \oplus \mathbf{a}_{12} \|} \frac{\gamma_{\ominus \mathbf{a}_2 \oplus \mathbf{a}_{23}} \| \ominus \mathbf{a}_2 \oplus \mathbf{a}_{23} \|}{\gamma_{\ominus \mathbf{a}_3 \oplus \mathbf{a}_{13}} \| \ominus \mathbf{a}_3 \oplus \mathbf{a}_{13} \|} = 1$$

$$(\text{see [2, pp 463]})$$

For further details we refer to the recent book of A.Ungar [2].

2. Main result

In this section, we present a proof of Smarandache's cevian triangle hyperbolic theorem in the Einstein relativistic velocity model of hyperbolic geometry. **Theorem 3** If $A_1B_1C_1$ is the cevian gyrotriangle of gyropoint P with respect to the gyrotriangle ABC, then

$$\frac{\gamma_{_{|PA|}|PA|}}{\gamma_{_{|PA_{1}|}|PA_{1}|}} \cdot \frac{\gamma_{_{|PB|}|PB|}}{\gamma_{_{|PB_{1}|}|PB_{1}|}} \cdot \frac{\gamma_{_{|PC|}|PC|}}{\gamma_{_{|PC_{1}|}|PC_{1}|}} = \frac{\gamma_{_{|AB|}|AB|} \cdot \gamma_{_{|BC|}|BC|} \cdot \gamma_{_{|CA|}|CA|}}{\gamma_{_{|AB_{1}|}|AB_{1}|} \cdot \gamma_{_{|BC_{1}|}|BC_{1}|} \cdot \gamma_{_{|CA_{1}|}|CA_{1}|}}.$$

Proof. If we use a theorem 2 in the gyrotriangle ABC (see Figure), we have

$$(1) \ \ \gamma_{_{|AC_1|}|AC_1|} \cdot \gamma_{_{|BA_1|}|BA_1|} \cdot \gamma_{_{|CB_1|}|CB_1|} = \gamma_{_{|AB_1|}|AB_1|} \cdot \gamma_{_{|BC_1|}|BC_1|} \cdot \gamma_{_{|CA_1|}|CA_1|}$$

If we use a theorem 1 in the gyrotriangle AA_1B , cut by the gyroline CC_1 , we get

$$(2) \quad \gamma_{|AC_1||AC_1|} \cdot \gamma_{|BC||BC|} \cdot \gamma_{|A_1P||A_1P|} = \gamma_{|AP||AP|} \cdot \gamma_{|A_1C||A_1C|} \cdot \gamma_{|BC_1||BC_1|}.$$

If we use a theorem 1 in the gyrotriangle BB_1C , cut by the gyroline AA_1 , we get

(3)
$$\gamma_{|BA_1||BA_1|} \cdot \gamma_{|CA||CA|} \cdot \gamma_{|B_1P||B_1P|} = \gamma_{|BP||BP|} \cdot \gamma_{|B_1A||B_1A|} \cdot \gamma_{|CA_1||CA_1|}$$

If we use a theorem 1 in the gyrotriangle CC_1A , cut by the gyroline BB_1 , we get

$$(4) \quad \gamma_{_{|CB_1|}|CB_1|} \cdot \gamma_{_{|AB|}|AB|} \cdot \gamma_{_{|C_1P|}|C_1P|} = \gamma_{_{|CP|}|CP|} \cdot \gamma_{_{|C_1B|}|C_1B|} \cdot \gamma_{_{|AB_1|}|AB_1|}.$$

We divide each relation (2), (3), and (4) by relation (1), and we obtain

(5)
$$\frac{\gamma_{|PA||PA|}}{\gamma_{|PA_1||PA_1|}} = \frac{\gamma_{|BC||BC|}}{\gamma_{|BA_1||BA_1|}} \cdot \frac{\gamma_{|B_1A||B_1A|}}{\gamma_{|B_1C||B_1C|}},$$

(6)
$$\frac{\gamma_{|PB||PB|}}{\gamma_{|PB_1||PB_1|}} = \frac{\gamma_{|CA||CA|}}{\gamma_{|CB_1||CB_1|}} \cdot \frac{\gamma_{|C_1B||C_1B|}}{\gamma_{|C_1A||C_1A|}},$$

(7)
$$\frac{\gamma_{|PC||PC|}}{\gamma_{|PC_1||PC_1|}} = \frac{\gamma_{|AB||AB|}}{\gamma_{|AC_1||AC_1|}} \cdot \frac{\gamma_{|A_1C||A_1C|}}{\gamma_{|A_1B||A_1B|}}.$$

Multiplying (5) by (6) and by (7), we have

$$\frac{\gamma_{_{|PA|}|PA|}}{\gamma_{_{|PA_{1}|}|PA_{1}|}} \cdot \frac{\gamma_{_{|PB|}|PB|}}{\gamma_{_{|PB_{1}|}|PB_{1}|}} \cdot \frac{\gamma_{_{|PC|}|PC|}}{\gamma_{_{|PC_{1}|}|PC_{1}|}} =$$

$$(8) \ \frac{\gamma_{|AB|}_{|AB|} \cdot \gamma_{|BC|}_{|BC|} \cdot \gamma_{|CA|}_{|CA|}_{|CA|}}{\gamma_{|A_1B|}_{|A_1B|} \cdot \gamma_{|B_1C|}_{|B_1C|} \cdot \gamma_{|C_1A|}_{|C_1A|}} \cdot \frac{\gamma_{|B_1A|}_{|B_1A|} \cdot \gamma_{|C_1B|}_{|C_1B|} \cdot \gamma_{|A_1C|}_{|A_1C|}_{|A_1C|}}{\gamma_{|A_1B|}_{|A_1B|} \cdot \gamma_{|B_1C|}_{|B_1C|} \cdot \gamma_{|C_1A|}_{|C_1A|}}$$

From the relation (1) we have

(9)
$$\frac{\gamma_{|B_1A||B_1A|} \cdot \gamma_{|C_1B||C_1B|} \cdot \gamma_{|A_1C||A_1C|}}{\gamma_{|A_1B||A_1B|} \cdot \gamma_{|B_1C||B_1C|} \cdot \gamma_{|C_1A||C_1A|}} = 1,$$

so

$$\frac{\gamma_{_{|PA|}|PA|}}{\gamma_{_{|PA_{1}|}|PA_{1}|}} \cdot \frac{\gamma_{_{|PB|}|PB|}}{\gamma_{_{|PB_{1}|}|PB_{1}|}} \cdot \frac{\gamma_{_{|PC|}|PC|}}{\gamma_{_{|PC_{1}|}|PC_{1}|}} = \frac{\gamma_{_{|AB|}|AB|} \cdot \gamma_{_{|BC|}|BC|} \cdot \gamma_{_{|CA_{1}|}|CA|}}{\gamma_{_{|AB_{1}|}|AB_{1}|} \cdot \gamma_{_{|BC_{1}|}|BC_{1}|} \cdot \gamma_{_{|CA_{1}|}|CA_{1}|}}.$$

References

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